International Journal of Multidiscipline

TLEP – International Journal of Multidiscipline
(Technology, Language, Education, and Psychology)
ISSN: 2488-9342 (Print) | 2488-9334 (Online)

Open Access | Peer-Reviewed | Monthly Publication | Impact factor: 8.497 / 2025

Methods For Construction Of The Zhegalkin Polynomial

Umarov Khabibullo Rakhmatullayevich
senior lecturer of the Department of Mathematics,
Gulistan State University, Uzbekistan, Gulistan
e-mail: umarovhr@mail.ru

Abstract
It is known that any Boolean function can be represented by a Zhegalkin polynomial (a polynomial modulo two), and this representation is unique up to a permutation of terms. Some of the most well-known methods for constructing such a polynomial have been given.
Keywords: DNF (disjunctive normal form), PKNF (perfect disjunctive normal form), sum modulo two, Pascal's triangle.

INTRODUCTION

The Zhegalkin polynomial (polynomial) is a polynomial whose coefficients are the numbers 0 or 1, and the conjunction and sum modulo 2, respectively, act as multiplication and addition operations. For example, for a Boolean function $f(x_1, x_2, x_3)$ of three variables x_1, x_2, x_3, the Zhegalkin polynomial will have the following form:

$$f(x_1,x_2,x_3)=a_0 \oplus a_1x_1 \oplus a_2x_2 \oplus a_3x_3 \oplus a_{12}x_1x_2 \oplus a_{13}x_1x_3 \oplus a_{12}x_1x_2 \oplus a_{23}x_2x_3 \oplus a_{123}x_1x_2x_3$$

The coefficients $a_0, a_1, \ldots, a_{123} \in \{0,1\}$, that is, can take values either 0 or 1, depending on what value the Boolean function takes on a particular set of variable values.

The polynomial was proposed in 1927 by Ivan Zhegalkin as a convenient means for representing Boolean logic functions.

The operation $\oplus$ (sum modulo 2) has other names, the Zhegalkin sum, the nonequivalence exclusive OR-NOT. Sometimes, for convenience, its notation uses the usual notation of addition, but you should not confuse it with disjunction, and even more so with the usual arithmetic operation of addition. The truth table of this operation has the form:

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

The sum $x \oplus y$ is true when one and only one of the constituent statements is true. If we compare the truth tables of the basic logical operations, we can see that $x \oplus y \to \overline{x \leftrightarrow y}$. That is, the Zhegalkin sum operation C is the negation of the equivalent.

For the two introduced operations $\oplus$, $\cdot$ (sum modulo two and conjunction), all logical laws are fulfilled:

1. Commutativity: $x \oplus y = y \oplus x$.
2. Associativity: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

that is, the result of $x \oplus y \oplus z$ does not depend on the arrangement of brackets.

3. Distributivity: $x \cdot (y \oplus z) = x \cdot y \oplus x \cdot z$.
4. $x \oplus x = 0$.
5. $0 \oplus x = x$.
6. $\bar{x} = x \oplus 1$.

Various methods can be used to construct the Zhegalkin polynomial: the method of indefinite coefficients, the Pascal triangle method, the DNF transformation and the PDNF transformation.

RESULTS

1. Method of undetermined coefficients

Let's find the Zhegalkin polynomial for the function $f(x_1,x_2,x_3)=(x_1 \cdot x_2 \vee x_3) \to \overline{x_2}$ using the method of indefinite coefficients. To do this, you first need to build a truth

ISSN: 2488-9342 (Print) | 2488-9334 (Online)
Open Access | Peer-Reviewed | Monthly Publication | Impact factor: 8.497 / 2025

table for the given Boolean function $f(x_1, x_2, x_3)$.

x_1	x_2	x_3	$x_1 \cdot x_2$	$x_1 \cdot x_2 \vee x_3$	$f(x_1,$
0	0	0	0	0	1
0	0	1	0	1	1
0	1	0	0	0	1
0	1	1	0	1	0
1	0	0	0	0	1
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	0

General view of the Zhegalkin polynomial for a function $f(x_1, x_2, x_3)$ of three variables x_1, x_2, x_3:

$$f(x_1,x_2,x_3) = a_0 \oplus a_1x_1 \oplus a_2x_2 \oplus a_3x_3 \oplus a_{12}x_1x_2 \oplus a_{13}x_1x_3 \oplus a_{12}x_1x_2 \oplus a_{23}x_2x_3 \oplus a_{123}x_1x_2x_3$$

We successively substitute sets of values of variables and find the coefficients $a_0, a_1, \ldots, a_{123} \in \{0,1\}$.

$f(0,0,0) = a_0 = 1$;

$f(0,0,1) = a_0 \oplus a_3 = 1 \Rightarrow 1 \oplus a_3 = 1 \Rightarrow a_3 = 0$;

$f(0,1,0) = a_0 \oplus a_2 = 1 \Rightarrow 1 \oplus a_2 = 1 \Rightarrow a_2 = 0$;

$f(0,1,1) = a_0 \oplus a_2 \oplus a_3 \oplus a_{23} = 0 \Rightarrow 1 \oplus 0 \oplus 0 \oplus a_{23} = 0 \Rightarrow 1 \oplus a_{23} = 0 \Rightarrow a_{23} = 1$; $f(1,0,0) = a_0 \oplus a_1 = 1 \Rightarrow 1 \oplus a_1 = 1 \Rightarrow a_1 = 0$;

$f(1,0,1) = a_0 \oplus a_1 \oplus a_3 \oplus a_{13} = 1 \Rightarrow 1 \oplus 0 \oplus 0 \oplus a_{13} = 1 \Rightarrow 1 \oplus a_{13} = 1 \Rightarrow a_{13} = 0$;

$f(1,1,0) = a_0 \oplus a_1 \oplus a_2 \oplus a_{12} = 0 \Rightarrow 1 \oplus 0 \oplus 0 \oplus a_{12} = 1 \Rightarrow 1 \oplus a_{12} = 0 \Rightarrow a_{12} = 0$;

$f(1,1,1) = a_0 \oplus a_1 \oplus a_2 \oplus a_{12} \oplus a_{13} \oplus a_{23} \oplus a_{123} = 0 \Rightarrow \Rightarrow 1 \oplus 0 \oplus 0 \oplus 0 \oplus 1 \oplus 0 \oplus 1 \oplus a_{123} = 0 \Rightarrow 1 \oplus a_{123} = 0 \Rightarrow a_{123} = 1$;

Substituting the found coefficients, we obtain the Zhegalkin polynomial:

$$f(x_1, x_2, x_3) = 1 \oplus x_1 \cdot x_2 \oplus x_2 \cdot x_3 \oplus x_1 \cdot x_2 \cdot x_3$$

.

2. Pascal's triangle method

Let's construct the Zhegalkin polynomial for the function from the previous method using Pascal's triangle.

x_1	x_2	x_3	$f(x_1, x$	Pascal's triangle	Terms
0	0	0	1	[1] 1 1 0 1 1 0 0	1
0	0	1	1	0 0 1 1 0 1 0	x_3
0	1	0	1	0 1 0 1 1 1	x_2
0	1	1	0	[1] 1 1 0 0	$x_2 \cdot x_3$
1	0	0	1	0 0 1 0	x_1
1	0	1	1	0 1 1	$x_1 \cdot x_3$
1	1	0	0	[1] 0	$x_1 \cdot x_2$
1	1	1	0	[1]	$x_1 \cdot x_2 \cdot x$

Let us explain how Pascal's triangle is filled. The top line of the triangle defines the value vector of the Boolean function $f(11101100)$. In each line, starting from the second, any element of such a triangle is calculated as the modulo 2 sum of two adjacent elements of the previous line. So, the elements of the second line: $1 \oplus 1 = 0$, $1 \oplus 1 = 0$, $1 \oplus 0 = 1$, $0 \oplus 1 = 1$, $1 \oplus 1 = 0$, $0 \oplus 0 = 0$. The elements of other rows are calculated similarly.

The left side of Pascal's triangle corresponds to the sets of values of the variables of the original function $f(x_1, x_2, x_3)$. Connecting the variables whose values in the set are equal to 1 with a conjunction sign, we get the term in the Zhegalkin polynomial. The set (000) corresponds to 1, and the set (001) corresponds to S , and so on.

Snce the terms 1, $x_2 \cdot x_3$, $x_1 \cdot x_2$, $x_1 \cdot x_2 \cdot x_3$, correspond to the units of the left side of the triangle, then the Zhegalkin polynomial:

$$f(x_1,x_2,x_3)=1\oplus x_1\cdot x_2\oplus x_2\cdot x_3\oplus x_1\cdot x_2\cdot x_3$$

.3. DNF transformation

Using the basic laws of the algebra of logic, we first reduce the function to a DNF.

$$f(x_1,x_2,x_3)=(x_1\cdot x_2\vee x_3)\to\overline{x_2}=\{\text{use the equivalence } x\to y=\overline{x}\vee y\}=$$
$$=\overline{x_1\cdot x_2\vee x_3}\vee\overline{x_2}=\{\text{use de Morgan's law } \overline{x\cdot y}=\overline{x}\vee\overline{y}\}==(\overline{x_1}\vee\overline{x_2})\cdot\overline{x_3}\vee\overline{x_2}=$$
$$=\{\text{use the distributive law } (x\vee y)\cdot z=x\cdot z\vee y\cdot z\}=$$
$$=\overline{x_1}\cdot\overline{x_3}\vee\underbrace{\overline{x_1}\cdot\overline{x_3}}_{\text{assimilate}}\overline{x_2}\vee\overline{x_2}=\overline{x_1}\cdot\overline{x_3}\vee\overline{x_2}$$

– DNF.

Further, in the resulting DNF, it is necessary to “get rid” of the disjunction using de Morgan’s laws:

$$\overline{x_1}\cdot\overline{x_3}\vee\overline{x_2}=\overline{\overline{\overline{x_1}\cdot\overline{x_3}}\cdot x_2}.$$

We replace each negation of $\overline{x}=x\oplus 1$ and apply the logical laws written above, we get:

$$\overline{\overline{\overline{x_1}\cdot\overline{x_3}}\cdot x_2}=1\oplus\overline{x_1}\cdot\overline{x_3}\cdot x_2=1\oplus(1\oplus(1\oplus x_1)\cdot(1\oplus x_3))\cdot x_2=$$
$$=1\oplus(1\oplus 1\oplus x_3\oplus x_1\oplus x_1\cdot x_3)\cdot x_2=1\oplus(x_3\oplus x_1\oplus x_1\cdot x_3)\cdot x_2=$$
$$=1\oplus x_2\cdot x_3\oplus x_1\cdot x_2\oplus x_1\cdot x_2\cdot x_3$$

– Zhegalkin polynomial.

4. Transforming PDNF To build a PDNF according to the truth table, we select sets on which the function f takes a value equal to 1. If the value of a variable in this set is 0, then it is taken with negation, if the value of the variable is 1, then the variable is taken without negation. Connecting the variables of the corresponding set with a conjunction sign, we obtain an elementary conjunction. Then the disjunction of all such elementary conjunctions is PDNF.

$$f(x_1,x_2,x_3)=\overline{x_1}\cdot\overline{x_2}\cdot\overline{x_3}\vee\overline{x_1}\cdot\overline{x_2}\cdot x_3\vee\overline{x_1}\cdot x_2\cdot\overline{x_3}\vee x_1\cdot\overline{x_2}\cdot\overline{x_3}\vee x_1\cdot\overline{x_2}\cdot x_3$$

.

To construct a Zhegalkin polynomial using PDNF, it is necessary to eliminate the operations of disjunction and negation, then open the brackets.

$$f(x_1,x_2,x_3)=\overline{x_1}\cdot\overline{x_2}\cdot\overline{x_3}\oplus\overline{x_1}\cdot\overline{x_2}\cdot x_3\oplus\overline{x_1}\cdot x_2\cdot\overline{x_3}\oplus x_1\cdot\overline{x_2}\cdot\overline{x_3}\oplus x_1\cdot\overline{x_2}\cdot x_3=$$
$$=(1\oplus x_1)\cdot(1\oplus x_2)\cdot(1\oplus x_3)\oplus(1\oplus x_1)\cdot(1\oplus x_2)\cdot x_3\oplus(1\oplus x_1)\cdot x_2\cdot(1\oplus x_3)\oplus$$
$$\oplus x_1\cdot(1\oplus x_2)\cdot(1\oplus x_3)\oplus x_1\cdot(1\oplus x_2)\cdot x_3=$$
$$=1\oplus x_1\oplus x_3\oplus x_2\oplus x_2\cdot x_3\oplus x_1\oplus x_1\cdot x_3\oplus x_1\cdot x_2\oplus x_1\cdot x_2\cdot x_3\oplus$$
$$\oplus x_3\oplus x_2\cdot x_3\oplus x_1\cdot x_3\oplus x_1\cdot x_2\cdot x_3\oplus x_2\oplus x_2\cdot x_3\oplus x_1\cdot x_2\oplus x_1\cdot x_2\cdot x_3\oplus x_1\oplus$$
$$\oplus x_1\cdot x_3\oplus x_1\cdot x_2\oplus x_1\cdot x_2\cdot x_3\oplus x_1\cdot x_3\oplus x_1\cdot x_2\cdot x_3=$$
$$=1\oplus x_2\cdot x_3\oplus x_1\cdot x_2\oplus x_1\cdot x_2\cdot x_3$$

– Zhegalkin polynomial.

Conclusion

Mathematics is a very exact science, but you can show your imagination in it by solving problems in various ways. Discrete mathematics is no exception to this.

Ivan Ivanovich Zhegalkin rendered a great service to mankind when he deduced a polynomial, later named after him. The polynomial, whose members are connected only by two operations and a unit, turned out to be incredibly useful and is very widely used by a person in the course of his life and work.

REFERENCES

Yablonsky S.V. Introduction to Discrete Mathematics. 1986.

Marchenkov S.S. Closed classes of Boolean functions. 2000.

Suprun V.P. Fundamentals of the theory of Boolean functions.2017.

Gavrilov G.P., Sapozhenko A.A. Problems and Exercises in Discrete Mathematics.2004.

Likhtarnikov L.M., Sukacheva T.G. Mathematical Logic.2009.

ЖАМУРАТОВ, К., УМАРОВ, Х.Р., & АЛИМБЕКОВ, А. Решение одной задачи движения грунтовых вод в области с подвижной границей при наличии испарения. НАУЧНЫЙ АЛЬМАНАХ Учредители: ООО" Консалтинговая компания Юком, 81-84.

Жамуратов, К., Умаров, Х., & Холбоев, С. (2016). Решение одной задачи теории фильтрации методом квазистационарного приближения. Вестник ГулГУ, (2016/2), 9.

Zhamuratov K. On filtration near new canals and reservoirs with a piecewise constant coefficient. Tashkent: IKsVTs AN UzSSR, 1979, issue. 54. p.100-109.

Umarov, X. R., & Asqarbekova, D. J. (2025). YIG‘INDI VA KO‘PAYTMALARNI HISOBLASHDA KOMPLEKS ANALIZ METODLARIDAN FOYDALANISH. МОЯ ПРОФЕССИОНАЛЬНАЯ КАРЬЕРА. Международная научно-образовательная электронная библиотека (НЭБ)«МОЯ ПРОФЕССИОНАЛЬНАЯ КАРЬЕРА», (68 (том 2)).

Narjigitov, X., Umarov, X. R., & Zulfiqorova, M. A. (2025). FUR'YE QATORI YIG ‘INDISINING AYRIM FUNKSIONAL XOSSALARI. Международная научно-образовательная электронная библиотека (НЭБ)«МОЯ ПРОФЕССИОНАЛЬНАЯ КАРЬЕРА», (75 (том 1)).

Umarov, X. R., & Boymurodov, D. I. (2025). GAMMA FUNKSIYANING AYRIM XOSSALARI. Международная научно-образовательная электронная библиотека (НЭБ)«МОЯ ПРОФЕССИОНАЛЬНАЯ КАРЬЕРА», (70 (том 1)).

Umarov, X. R., & Abduraximova, D. D. (2025). MATEMATIKADAN OLIMPIADA MASALALARINI YECHISHDA MATEMATIK ANALIZ METODLARIDAN FOYDALANISH. МОЯ ПРОФЕССИОНАЛЬНАЯ КАРЬЕРА. Международная научно-образовательная электронная библиотека (НЭБ)«МОЯ ПРОФЕССИОНАЛЬНАЯ КАРЬЕРА», (68 (том 2)).

Жамуратов, К., Умаров, X., & Бойкузиева, М. (2025). К ПОСТРОЕНИЮ МАТЕМАТИЧЕСКОЙ МОДЕЛИ ОДНОЙ ЗАДАЧИ ДВИЖЕНИЯ ГРУНТОВЫХ ВОД ВБЛИЗИ НОВЫХ ВОДОХРАНИЛИЩ И КАНАЛОВ. Международная научно-образовательная электронная библиотека (НЭБ)«МОЯ ПРОФЕССИОНАЛЬНАЯ КАРЬЕРА», (70 (том 1)).

Zhamuratov, K., Umarov, K., & Dodobayev, A. (2024, May). Drainage of a semi-infinite aquifer in the presence of evaporation. In AIP Conference Proceedings (Vol. 3147, No. 1). AIP Publishing.

Жамуратов, К., Умаров, X. Р., & Турдимуродов, Э. М. (2024). О решении методом регуляризации одной системы функциональных уравнений с дифференциальным оператором (Doctoral dissertation, Белорусско-Российский университет) (Doctoral dissertation, Doctoral dissertation, Белорусско-Российский университет).

Агафонов, А., Умаров, X., & Душабаев, О. (2023). ДРЕНИРОВАНИЕ ПОЛУ БЕСКОНЕЧНОГО ВОДОНОСНОГО ГОРИЗОНТА ПРИ НАЛИЧИИ ИСПАРЕНИЯ. Евразийский журнал технологий и инноваций, 1(6 Part 2), 99-104.

4. Narjigitov, X., Jamuratov, K., Umarov, X., & Xudayqulov, R. (2023). SEARCH PROBLEM ON GRAPHS IN THE PRESENCE OF LIMITED INFORMATION ABOUT THE SEARCH POINT. Modern Science and Research, 2(5), 1166-1170.

5. Умаров, X. Р., & Жамуратов, К. (2015). Решение задачи о притоке к математическому совершенному горизонтальному дренажу. Актуальные направления научных исследований XXI века: теория и практика, 3(8-4), 303-307.